SS 2011 22. Juni 2011

Exercises for the Lecture Logics
Sheet 9

Prof. Dr. Klaus Madlener Delivery until 29. Juni 2011 10:00 Uhr

Exercise 1: [Validity, tutorial]
Prove that the following formulas are universally valid:

$A_1 \equiv (a = 3 \to (q \to \forall y[a = y \to y = 4])) \to ((a = 3 \to q) \to (a = 3 \to \forall y[a = y \to y = 4]))$

$A_2 \equiv \forall z \forall x[p(x)] \to p(f(a, 5))$
$A_3 \equiv \exists P \forall Q[P \to (Q \wedge r) \to P \vee r]$
$A_4 \equiv \exists P[P]$

Exercise 2: [Substitution, tutorial]

1. Find a formula A, a term t and a variable x such that the substitution $A_x[t]$ is allowed, $A_x[t]$ is universally valid, but A is not universally valid.
2. Let σ be the substitutionb from exercise 5. Apply this substitution to the following formulas and terms:
   $A_1 \equiv (x < 3 \to p(x_1))$
   $A_2 \equiv \exists x[x = 0 \vee P(x_3)]$
   $A_3 \equiv \forall x[\neg x_1 = 0]$
   $A_4 \equiv \forall x_1[\neg x_1 = 0] \to p$

Exercise 3: [semantic conclusion, tutorial]
Let $A \in$ Form. Prove or disprove:

1. $\exists y \forall x\ A \models \forall x \exists y\ A$
2. $\forall x \exists y\ A \models \exists y \forall x\ A$
3. $\forall x\ f(x) = g(x) \models f = g$.

Exercise 4: [Tautologies in PL]
Prove that the following formulas are universally valid:

$A_1 \equiv \forall x \exists P[P(x) \vee x = f(a)] \to (Q(y, z) \to \forall x \exists P[P(x) \vee x = f(a)])$
$A_2 \equiv \forall x[q(x)] \to q(h(g(a, f(b)), b, f(c)))$
$A_3 \equiv \forall z[\neg(x = f(x) \to p(f(x))) \to (p(f(x)) \to x = f(x))]$
$A_4 \equiv \exists P[P \to q \vee r]$

Exercise 5: [Substitution, 10 P]
Let the substitution σ be defined by

$$\begin{aligned}
\sigma(x_1) &= x + 3 \cdot x \\
\sigma(x_2) &= 3 - (x + x_1) \cdot 2 \\
\sigma(x_3) &= 42 \\
\sigma(x_4) &= f(a, g(b)) \\
\sigma(x_5) &= if\ (x > 3)\ then\ 5\ else\ 3 \\
\sigma(x_6) &= g(y * 2).
\end{aligned}$$

Apply σ to the following formulas. Check whether the substitution is allowed.
$A_1 \equiv x_1 \geq x_3$
$A_2 \equiv \forall x[x = 42 \rightarrow \neg(x_4 = 3)]$
$A_3 \equiv \exists y[f(y) = 0 \rightarrow \forall x[x \geq x_2]]$
$A_4 \equiv p(x_1) \vee \forall x[x + 3 > x_6]$
$A_5 \equiv \forall x[x_5 = 5 \rightarrow x > 3]$
$A_6 \equiv x_3 < x_4 \vee \forall y[p(y) \vee y = 3]$
$A_7 \equiv \forall x_3[x_3 = 42]$

Exercise 6: [Decidability of equational logic, 5P]
We know from the lecture that the validity of a formula is generally undecidable. However, this does not hold for all restricted classes of formulas.

Consider formulas of *pure equational logic*, i.e. formulas with the operators $\{\neg, \wedge, \vee, \forall, \exists\}$, that contain only variables "x_i" and „=“For example:

$$\forall x \forall y \exists z\ [(x = y \wedge y \neq z) \rightarrow x = z].$$

Sketch a procedure that decides whether such a formula is universally valid. Argue for the correctness of your procedure.

Delivery: until 29. Juni 2011 10:00 Uhr into the box next to room 34-401.4