Exercise 26: from Meseguer/Goguen

Let $\Sigma = sig_1$, `sorts N,E`; `functions` $0 :\to N, s : N \to N, 1 :\to E, f : N \to E$

Assumption: There is a specification $D = (\Sigma, E)$ with finite E, such that $T := T_D \cong \mathfrak{A}_1$. Wlog E does not contain trivial equations $t = t$.

Let $E = E_1 \cup E_2 \cup E_3$

1. E_1 contains only ground equations
2. Equations from E_2 match the pattern $l[x] = r$, $l = r[x], l[x] = r[y]$, (where $l[x]$ means that variable x occurs in the term l). I.e. no equation in E has the same variable at both sides.
3. Equations from E_3 match the pattern $l[x] = r[x]$, i.e. at least one variable occurs on both sides.

We prove now, that $E_2 = \emptyset$, $E_3 = \emptyset$, and E_1 must be infinite.

1. Claim: $E_2 = \emptyset$: Look at the cases $l[x] = r$, $l = r[x], l[x] = r[y]$
   - $l[x] = r \in E_2$: Then $T_N = V_N \cup \{s^n0 \mid n \in \mathbb{N}\} \cup \{s^n x \mid n \in \mathbb{N}\}$ and $T_E = V_E \cup \{1\} \cup \{ft \mid t \in T_N\}$. We have two cases: $x \in V_N$ or $x \in V_E$:
     a) $x \in V_N$: Then $l[x] \equiv s^n x$ and $r \equiv s^m 0$, or $l[x] \equiv fs^n x$ and $r \in \{1, fs^m 0\}$. But we clearly know, that $\mathfrak{A}_1$ is no model for the formulas $\forall x s^n x = s^m 0$, $\forall x f s^n x = 1$, and $\forall x f s^n x 0 f s^m 0$
     b) $x \in V_E$: The $l[x] \equiv x$ and $r \in \{1, fs^m 0\}$. Again we clearly know that $\forall x x = 1$ and $\forall x x = fs^m 0$ do not hold.

     We conclude that E_2 does not contain equations of the form $l[x] = r$
   - $l = r[x]$: analogous to the previous case
   - $l[x] = r[y]$: Look at arbitrary ground terms substituted for y, we obtain $r' = l[x]$. We know $\mathfrak{A}_1 \models \forall x, y l[x] = r[y]$ implies $\mathfrak{A}_1 \models \forall x l[x] = r'$. The latter does not hold as seen in the first case.

     We conclude that E_2 does not contain equations of the form $l[x] = r[y]$

   Together we know $E_2 = \emptyset$
2. Similarly we prove that E_3 is empty. The main problem is to identify the cases, where a variable may occur twice.
3. Now to E_1: Let $l = r \in E_1$, then $l \equiv s^n 0$ and $r \equiv s^m 0$ or $l \in \{fs^n 0, 1\}$ and $r \in \{fs^m 0, 1\}$.

   If $\mathfrak{A}_1 \models s^n 0 = s^m 0$ then $n = m$ and we have a trivial equation.

   If $\mathfrak{A}_1 \models fs^n 0 = fs^m 0$ then only for $n \neq m$ we have a non-trivial equation, with both n, m either even or odd.

   If $\mathfrak{A}_1 \models fs^n 0 = 1$ then n must be odd.

Now assume E_1 to be finite. Then $E_1 = \{fs^{n_i} = fs^{m_i} \mid i = 1, \dots k\} \cup \{fs^{l_i} = 1 \mid i = 1, \dots k'\}$. Let $q = 1 + \max(\{l_i\} \cup \{m_i\} \cup \{n_i\})$. Then we know $E_1 \nvdash fs^q 0 = 1$ and $E_1 \nvdash fs^q 0 = s^q 0$. Since $E_2 = \emptyset$ and $E_3 = \emptyset$ we know $\mathfrak{A}_1 \ncong T$ for finite E_1.