SS 2011 06. Juli 2011

Exercises for the Lecture Logics
Sheet 11

Prof. Dr. Klaus Madlener Delivery until 13. Juli 2011 10:00 Uhr

Exercise 1: [Axiomatisation, tutorial]

1. Define a first-order-formula A_n, such that every interpretation satisfying A_n has exactly n elements. More precisely, in every interpretation satisfying A_n, the domain D has exactly n elements.
2. Define a first-order-formula A_∞, such that every satisfying interpretation of A_∞ has infinitely many elements.
3. Prove that the compactness theorem does not hold for second order predicate logic.

Exercise 2: [Deductions in $\mathcal{F}$, 2+2P]
Prove:

1. $\forall x[p(x,y)],\ y = z \vdash_{\mathcal{F}} \forall x[p(x,z)]$.

2. $\forall x[p(x) \to q(x)], \forall x[p(x)] \vdash_{\mathcal{F}} q(f(a))$

Exercise 3: [Soundness of $\mathcal{F}\prime$, 4+1P]

1. Prove that the generalisation rule is sound.
2. As mentioned in the lecture, the proposition $\Sigma \vdash_{\mathcal{F}\prime} A \rightsquigarrow \Sigma \vdash_{\mathcal{F}} A$ *does not generally hold*. This means that there are conclusions from Σ which can be deduced in $\mathcal{F}\prime$, but not in $\mathcal{F}$. Why does this result not contradict the fact that both systems are sound?

Exercise 4: [Theories, 3+3P]
Prove:

1. Let M be a first-order-theory. There is an interpretation I that satisfies M, iff M is consistent.
2. If T is a consistent, incomplete first-order-theory, then for every closed formula A with $A, \neg A \notin \Sigma$, both $T_{T\cup\{A\}}$ and $T_{T\cup\{\neg A\}}$ are consistent theories.
3. Let T_1 and T_2 be first-order theories. If $T_1 \subsetneq T_2$ and T_1 complete, then T_2 inconsistent.

Exercise 5: [Theories, 5P]
Let T be a consistent, incomplete first-order-theory. Prove that there are at least two different relational structures satisfying T.

Exercise 6: [Non standard models, 5P]
Prove that there are non-standard-models for the Peano axioms (slide 220). I.e. prove that there is an interpretation that satisfies the Peano axioms but that is not isomorphic to $\mathbb{N}$.

Hint: Consider the following extended axiom system and apply the compactness theorem:

$$P^* := P \cup \{A_i \mid i \in \mathbb{N}\},$$

where $A_i \equiv \exists z\ [S^i(0) + z = \infty]$. ∞ is a new constant and $S^i(0)$ is the i-fold application of S to 0. I.e. $S^3(0) \equiv S(S(S(0)))$.

Exercise 7: [Axiomatisation, 1+3+5P]
Characterise propositional logic (with the operators $\neg$, $\wedge$, and $\vee$ and with the constants $true$ and $false$) using predicate logic. Do the following:

1. Find a suitable language of predicate logic, s.t. every term represents a boolean formula.
2. Find Axioms that characterise the boolean operators. I.e. if two terms t_1 and t_2 are equivalent in propositional logic and if an interpretation I satisfies your axioms, then $I(t_1) = I(t_2)$ must hold.
3. Find axioms for the predicate constants $taut(x)$, $uns(x)$, $conc(x)$, and $eq(x, y)$ such that
   - $\Sigma \models taut(t)$ iff t is a propositional tautology.
   - $\Sigma \models uns(t)$ iff t is unsatisfiable in propositional logic.
   - $\Sigma \models conc(t_1, t_2)$ iff $t_1 \models t_2$ holds in propositional logic.
   - $\Sigma \models eq(t_1, t_2)$ iff t_1 and t_2 are equivalent in propositional logic.

   Σ is the set of your axioms in (2) and (3). Argue for the soundness of your axioms. How would you formally prove the soundness?

Delivery: until 13. Juli 2011 10:00 Uhr into the box next to room 34-401.4